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The Adjacency Energy of a T_2 Hypergraph

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Abstract

A hypergraph $H = (X, D)$ is said to be a T_2 hypergraph if for any three distinct vertices u, v and w in$H$, there exist a hyperedge containing u, v but not w and another hyperedge containing w but not u and$v$. In this article, the adjacency energy of a T_2 hypergraph is studied. It is shown that, $\epsilon(H) > \sqrt{\frac{w(H)}{n}}$.

Keywords: T_2 hypergraph, Adjacency matrix, Adjacency energy.
Subject Classification: 05C65

1 Introduction

The basic definitions and terminalogies of a hypergraph are not given here and we refer it [3] and [13]. The concept of hypergraph was introduced by Berge in 1967. Later the same concept was studied by different authors in [13] and [1], Seena V and Raji Pilakkat were introduced Hausdroff hypergraph, T_0 hypergraph and T_1 hypergraph. Based on [10] and [11] we introduced a new class of hypergraph namely T_2 hypergraph is studied the parameter adjacency energy of a hpergraph. Throughout this article $H = T_2$ is a simple connected hypergraph with order n and size m. Here the order and size are the minimum number of vertices and edges used to define a T_2 hypergraph. The following definitions and theorems are used in sequel.

Definition 1.1. *[9] A hypergraph $H = (X, D)$is said to be a Hausdroff hypergraph if for any two distinct vertices u, v of X there exist hyperedges D_1 and $D_2 \in D$ such that $u \in D_1$,$v \in D_2$ and $D_1 \cap D_2 = \phi$.*

Definition 1.2. *[10] A hypergraph $H = (X, D)$is said to be a T_0 hypergraph if for any two distinct vertices u and v of X there exist a hyperedge containing one of them but not the other.*

Definition 1.3. *[11] A hypergraph $H = (X, D)$ is said to be a T_1 hypergraph if for any two distinct vertices u andv of X there exist a hyperedge containing u but not v and another hyperedge containing v but not u.*

Definition 1.4. *[2] The Wiener index $W(H)$ is defined by $W(H) = \sum\limits_{u,v \in X(H)} d_H(u, v)$.*

Definition 1.5. *[12] The Gutman index is defined as $GutH = \sum\limits_{u,v \in X(H)} d_H(u, v)d_H(u)d_H(v)$*

Definition 1.6. *[6] The rank of a hypergraph is the maximum cardinality of any of the edges in the hypergraph.*

Definition 1.7. *[6] The number of edges of a hypergraph H that are incident to a given vertex is called the degree of the vertex.*

Definition 1.8. *[7] The adjacency matrix is the square matrix which rows and columns are indexed by the vertices of H and where for all $u, v \in X, u \neq v, a_{uv} = |\{d \in D/u, v \in D\}|$ and $a_{uu} = 0$.*

Theorem 1.9. [4] *For a graph G on n vertices and m edges, $E(G) \leq \frac{2m}{n} + \sqrt{(n-1)[2m - (\frac{2m}{n})^2]}$.*

Result 1.10. (i) *The minimum number of edges need to define a T_2 hypergraph is $m=\left[\frac{2n+5}{4}\right]$ where n is the number of vertices.*

(ii) *For a T_2 hypergaph H, the minimum degree $\delta(H) = 2$.*

(iii) *For a T_2 hypergraph H, rank $r = \left[\frac{2n+1}{4}\right]$ where $n \geq 3$.*

2 T_2 Hypergraph

Based on T_0 and T_1 hypergraphs, a class of hypergraph namely T_2 hypergraph is defined in the following way and is observed some of its properties.

Definition 2.1. *A hypergraph $H = (X, D)$ is said to be a T_2 hypergraph if for any three distinct vertices u, v and w in H there exist a hyperedge containing u and v but not w and another hyperedge containing w but not u and v.*

Result 2.2. (i) *The minimum number of edges need to define a T_2 hypergraph is $m=\left[\frac{2n+5}{4}\right]$ where n is the number of vertices.*

(ii) *For a T_2 hypergaph H, the minimum degree $\delta(H) = 2$.*

(iii) *For a T_2 hypergraph H, rank $r = \left[\frac{2n+1}{4}\right]$ where $n \geq 3$.*

3 The adjacency energy of a T_2 hypergraph

Adjacency energy of a hypergraph is introduced by Kaue Cardoso and Vilmar Trevisan in [6]. The definition is studied from [7]. In this section we find the Adjacency energy of a T_2 hypergaph.

Theorem 3.1. *If $H = (X, D)$ is a T_2 hypergraph with n vertices and m edges, then*

$$\epsilon(H) > \sqrt{\delta \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2}.$$

Proof. $(\epsilon(H))^2 = (\sum_{i=1}^{n} \lambda_i)^2 \geq \sum_{i=1}^{n} \lambda_i^2 > 2\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2 > \delta \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2$.

Thus $\epsilon(H) > \sqrt{\delta \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2}$. □

Theorem 3.2. *Let $H = (X, D)$ be a T_2 hypergraph with maximum degree Δ. If $n \geq 5$ then $\epsilon(H) \leq \sqrt{n(\Delta - 1)G(H)} < n\sqrt{(\Delta^2 - \Delta)W(H)}$ equality hold only if H is a T_2 hypergraph with n=3 and $\Delta = 2$.*

Proof. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of $A(H)$, then

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2 \leq \sum_{i=1}^{n} \left[d(i)^2 + \sum_{j=1}^{m} (a_{ij})^2\right]$$

$$\leq (\Delta - 1) \sum_{i=1}^{n} d(i)d(j)d(i,j) \leq (\Delta - 1)G(H).$$

Now $\epsilon(H)^2 = \sum_{i=1}^{n} \lambda_i^2 \leq n \sum_{i=1}^{n} \lambda_i^2 \leq n(\Delta - 1)G(H)$.

$$G(H) < n\Delta W(H)$$

$$G(H) \leq \sqrt{n(\Delta - 1)G(H)} < \sqrt{n(\Delta - 1)n\Delta W(H)}$$

$$G(H) \leq \sqrt{n(\Delta - 1)G(H)} < n\sqrt{(\Delta^2 - \Delta)W(H)}.$$

□

Theorem 3.3. *Let H be a T_2 hypergraph with maximum degree Δ and $|v| = n$ then* $\epsilon(H) \leq \lambda_1 + \sqrt{(n-1)\left[(\Delta-1)G(H) - \lambda_1^2\right]}$.

Proof. We have $\sum_{i=1}^{n} \lambda_i^2 = \lambda_1^2 + \sum_{i=2}^{n} \lambda_i^2$ and so $\sum_{i=1}^{n} \lambda_i^2 \leq (\Delta-1)G(H)$
$\epsilon(H) = \lambda_1 + \sum_{i=2}^{n} \lambda_i^2$.

This gives that $\epsilon(H) - \lambda_1 = \sum_{i=2}^{n} \lambda_i^2$.

$$\begin{aligned}
(\epsilon(H) - \lambda_1)^2 &\leq (\sum_{i=2}^{n} \lambda_i^2)^2 \\
&\leq (n-1)\sum_{i=2}^{n} \lambda_i^2 \\
&= (n-1)(\Delta-1)G(H) - \lambda_1^2 \\
\epsilon(H) - \lambda_1 &\leq \sqrt{(n-1)(\Delta-1)G(H) - \lambda_1^2} \\
\epsilon(H) &\leq \lambda_1^2 + \sqrt{(n-1)(\Delta-1)G(H) - \lambda_1^2}.
\end{aligned}$$

□

Theorem 3.4. *If H is a T_2 hypergraph with rank r and maximum degree Δ then $\epsilon(H) \geq \sqrt{n(\Delta-1)r}$ equality hold if H is a hypergraph with $\Delta = m = n = 3$.*

Proof. From lemma (25) in [6]

$$\begin{aligned}
(\epsilon(H))^2 &\geq 2(\sum_{i=1}^{n} \lambda_i^2) \\
&\geq 2\sum_{i=1}^{n}\sum_{j=1}^{m}(a_{ij})^2 \\
&\geq 2\sum_{i=1}^{n}\sum_{j=1}^{m}(a_{ij}) \\
&\geq n(\Delta-1)r \\
\epsilon(H) &\geq \sqrt{n(\Delta-1)r}.
\end{aligned}$$

□

Theorem 3.5. *If H is a T_2 hypergraph then* $\epsilon(H) \geq \sqrt{(\sum_{i=1}^{n} \lambda_i^2) + (\frac{16m^2-48m+35}{4})(detA)^{|\frac{4}{4m-5}|}}$ *equality hold if m=3.*

Proof. The result immediately follows from (lemma30) in [6]. □

Theorem 3.6. *Let H be a T_2 hypergraph with rank r and $|v| = n$ then $\epsilon(H) \geq \sqrt{\frac{(r-1)^2}{n}W(H)}$ equality hold if H is a 3-uniform hpergraph with one edge.*

Proof. From (lemma25) in [6]

$$\begin{aligned}\epsilon(H)^2 &\geq 2\sum_{i=1}^{n}\lambda_i^2\\ &\geq \sum_{i=1}^{n}\lambda_i^2\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_{ij})^2\\ &\geq \sum_{i=1}^{n}|\frac{1}{n}(\sum_{i=1}^{m}(a_{ij})^2|\\ &\geq 4\frac{(r-1)^2}{n}W(H)\\ &\geq 2\sqrt{\frac{(r-1)^2}{n}W(H)}\end{aligned}$$

□

Theorem 3.7. *Let H be a T_2 hypergraph with rank r and $|v| = n$ then $\epsilon(H) \geq \sqrt{\frac{(r-1)^2}{n}W(H)}$ equality hold if H is a 3-uniform hpergraph with one edge.*

Proof. From (lemma25) in [6]

$$\begin{aligned}\epsilon(H)^2 &\geq 2\sum_{i=1}^{n}\lambda_i^2\\ &\geq \sum_{i=1}^{n}\lambda_i^2\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_{ij})^2\\ &\geq \sum_{i=1}^{n}|\frac{1}{n}(\sum_{i=1}^{m}(a_{ij})^2|\\ &\geq 4\frac{(r-1)^2}{n}W(H)\\ &\geq 2\sqrt{\frac{(r-1)^2}{n}W(H)}\end{aligned}$$

□

Theorem 3.8. *Let H be a T_2 hypergraph with minimum degree δ and $|v|$=n then $\epsilon(H)^2 > \frac{W(H)}{n}$.*

Proof. From (lemma25) in [6]

$$\begin{aligned}\epsilon(H)^2 &\geq 2\sum_{i=1}^{n}\lambda_i^2\\ &\geq \sum_{i=1}^{n}\lambda_i^2\\ &> \sum_{j=1}^{n}[|\frac{1}{m(m-1)}(\sum_{i=1}^{n}a_{ij})^2|]\\ &> \frac{W(H)}{n}.\end{aligned}$$

□

Lemma 3.9. *Let H be a T_2 hypergraph with $n \geq 4$ and m edges then* $\rho(A) \geq \frac{2n+13}{2(n-1)}$

Theorem 3.10. *Let H be a T_2 hypergraph with $n \geq 4$ then* $\epsilon(H) \geq \frac{2n+13}{2(n-1)} + (n-1) + \ln|detA| - \ln\frac{2n+13}{2(n-1)}$ *equality hold if H is a 3-uniform hypergraph with one edge.*

Proof. We have

$$\begin{aligned}\epsilon(H) &= \sum_{i=1}^{n} \lambda_i \\ &= \lambda_1 + \sum_{i=2}^{n} \lambda_i \\ &\geq \lambda_1 + (n-1) + \sum_{i=2}^{n} \ln|\lambda_i| \\ &= \lambda_1 + (n-1) + \ln\prod_{i=2}^{n} \lambda_i \\ &= \lambda_1 + (n-1) + \ln(detA) - \ln\lambda_1.\end{aligned}$$

From lemma 3.9

$$\epsilon(H) \geq \frac{2n+13}{2(n-1)} + (n-1) + \ln|detA| - \ln\frac{2n+13}{2(n-1)}.$$

□

Theorem 3.11. *Let H be a T_2 k-graph on n vertices and m edges. Then*

$$\epsilon(H) =\leq \rho(A) + \sqrt{(n-1)(nm - \rho(A)},$$

where $\rho(A)$ is the spectral radius of the adjacency matrix.

Proof. In $T_2 - k$-graph, by using Cauchy Schwart inequality

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i &\leq nm \\ \rho(A) + \sum_{i=2}^{n} \lambda_i &\leq nm \\ \sum_{i=2}^{n} \lambda_i &\leq nm - \rho(A) \\ \epsilon(H) &\leq \rho(A) + \sum_{i=2}^{n} \lambda_i \\ &\leq \rho(A) + \sqrt{(n-1)\sum_{i=2}^{n} \lambda_i} \\ &\leq \sqrt{\rho(A) + (n-1)(nm - \rho(A))}.\end{aligned}$$

□

Lemma 3.12. *Let H be a connected T_2 k-graph and $A(H)$ its adjacency matrix. The hypergraph H is regular iff* $x = (1\ 1\ 1 \ldots\ 1)^T$

Theorem 3.13. Let H be a connected T_2 k-graph and A(H) its adjacency matrix then $\rho(A) \leq \frac{KW(H)}{\sqrt{n}}$ equality holds if H is regular.

Proof. In T_2 k-graph

$$\begin{aligned}\rho(A) &= \sqrt{\rho(A)^2} \\ &\leq \sqrt{X^T A^2 X} \\ &\leq \sqrt{\frac{(K\sum d(u,v))^2}{n}} \\ &= \frac{KW(H)}{\sqrt{n}}.\end{aligned}$$

□

Remark 3.1. *Let H be a T_2 hypergraph with rank r, minimum degree δ and $|v| = n$ then*

(i) $\sum_{i=1}^{n} \lambda_i^2 \geq n(r-1)\delta(H)$

(ii) $\sum_{i=1}^{n} \lambda_i^2 \geq \frac{(r-1)^2}{3n} W(H)$.

(iii) $\sum_{i=1}^{n} \lambda_i^2 \geq \frac{m}{m-1}\lambda_1$.

(iv) $\sum_{i=1}^{n} \lambda_i^2 \geq \frac{nm}{n+m-1}\lambda_1$.

Theorem 3.14. *If H is a T_2 hypergraph then*
$\epsilon(H) \geq \sqrt{(n(r-1))\delta(H) + (\frac{16m^2-48m+35}{4})(detA)^{|\frac{4}{4m-5}|}}$.

Proof. Use theorem 3.5 and remark 3.1 □

4 Conclusion

In this article, the adjacency energy of a T_2 hypergraph is obtained by using various graph parameters such as size m, rank r, Wiener index, Gutman index, maximum degree, minimum degree, spectral radius etc. In a similar way we can use other parameter also for calculating the adjacency energy of a T_2 hypergraph.

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